Note on the Double Periodicity of the Elliptic Functions.

By F. Franklin.

If a function w be defined by the equation

$$z = \int^{w} \frac{dw}{\sqrt{1 - w^2}},$$

whence

$$dw = \sqrt{1 - w^2}\, dz, \tag{1}$$

it follows geometrically from the definition that w is a singly periodic function of z, with a real period, viz. 2π. For if the complex variables z and w be represented by points in the usual way, equation (1), which may be written

$$dw = i\sqrt{w - 1}\sqrt{w + 1}\, dz,$$

shows that when z describes a straight line parallel to the axis of X, the tangent to the path of w makes with the axis of X an angle equal to $\frac{1}{2}\pi$ plus half the sum of the angles made with the axis of X by the lines drawn from w to 1 and -1; in other words, the normal to the path of w bisects internally the angle formed by the lines drawn from the point w to the points 1 and -1. Hence when z describes a straight line parallel to the axis of X, w describes an ellipse having the points 1 and -1 as foci; and therefore w returns periodically to a given value as z receives a certain increment ω. The value of ω is at once obtained, if, in equation (1), we attend to the moduli, instead of the arguments, of the factors. If we denote by s' the length of the path described by z, by s the corresponding length for w, and by r_1 and r_2 the focal radii of the point w, equation (1) gives

$$ds = \sqrt{r_1 r_2}\, ds',$$

whence

$$s' = \int ds / \sqrt{r_1 r_2}. \tag{2}$$

This integral, taken along the whole ellipse, gives the value of ω.

Likewise, when z moves parallel to the axis of Y, w describes a hyperbola having the points 1 and -1 as foci; and it is plain from (2) that z must describe

an infinite distance in order that w shall go off to infinity on the hyperbola. Hence there is no pure imaginary period.

Since to two points w on the same hyperbola belong two values of z which differ by a pure imaginary, the period ω, which is real, must be the same for all the ellipses; but the limit of these ellipses is a circle of infinite radius, for which we have obviously, from (2),

$$\omega = \int_0^{2\pi} d\vartheta = 2\pi.$$

This value of the integral of $ds/\sqrt{r_1 r_2}$ taken along any ellipse, may also be obtained from the consideration that the mean proportional between the focal radii of a point is equal to the conjugate diameter; then the result follows by projection from the circle.

From the fact that an ellipse is cut in only two points by one branch of a confocal hyperbola, it follows that there cannot be more than two values of z, noncongruous with respect to the modulus 2π, belonging to a given value of w. Hence, *à fortiori*, there can be no other period.

If, to fix the ideas, we suppose the lower limit in the integral

$$z = \int^w dw/\sqrt{1-w^2}$$

to be 0, and write $w = \phi(z)$, we have $\phi(0) = 0$; and it is plain from the symmetry of the curves that $\phi(-z) = -\phi(z)$ and $\phi(\pi + z) = -\phi(z)$, so that $\phi(\pi - z) = \phi(z)$; thus the two values of z corresponding to a given value of w are z and $\pi - z$.

If, instead of the equation $dw = \sqrt{1-w^2}\, dz$, we take

$$dw = \sqrt{(\alpha_1 - w)(\alpha_2 - w)}\, dz,$$

where α_1 and α_2 are any complex quantities, it is plain that when z describes a straight line parallel to the axis of X, w describes an ellipse whose foci are at α_1 and α_2; and when z describes a straight line parallel to the axis of Y, w describes a confocal hyperbola. Hence, as before, w is a singly periodic function with the real period 2π. If $dw = \sqrt{(w-\alpha_1)(w-\alpha_2)}\, dz$, the ellipses are described when z moves parallel to the axis of X, the hyperbolas when z moves parallel to the axis of Y; so that w has the pure imaginary period $2\pi i$.

To go back to a still simpler case: the path of the function w defined by the equation $z = \int^w dw/w$, i. e.

$$dw = w\,dz$$

is obviously a straight line through the origin when z moves parallel to the axis of X, and a circle with its centre at the origin when z moves parallel to the axis of Y. Hence w is a singly periodic function with the period $2\pi i$.

The following theorem was given by Clifford without demonstration,* and has been proved by a special method by Crofton (London Math. Soc. Proc. (1867), II, 37): *If* 1, 2, 3, 4 *be four concyclic foci of a bicircular quartic through the point* P, *the tangent to the quartic at* P *bisects the angle between the circles* P_{12}, P_{34}. This property may be exhibited in a form which makes the double periodicity of the function defined as the inverse of an elliptic integral of the first kind follow geometrically from its definition, in the same way in which the single periodicity of the function inverse to $\int dw/\sqrt{(\alpha_1 - w)(\alpha_2 - w)}$ follows from the like property for the ellipse and hyperbola. Namely, if we denote by ϑ_{AB} the angle made with an arbitrary line of reference by the line joining any two points A and B, and by ϑ_t, $\vartheta_{t'}$, $\vartheta_{t''}$, the angles made with the line of reference by the tangents at P to the quartic and the two circles respectively, we have, by elementary geometry,

$$\vartheta_{t'} + \vartheta_{12} = \vartheta_{P1} + \vartheta_{P2}, \quad \vartheta_{t''} + \vartheta_{34} = \vartheta_{P3} + \vartheta_{P4};$$

but, by Clifford's theorem,

$$\vartheta_t = \frac{1}{2}(\vartheta_{t'} + \vartheta_{t''});$$

hence

$$\vartheta_t + \frac{1}{2}(\vartheta_{12} + \vartheta_{34}) = \frac{1}{2}(\vartheta_{P1} + \vartheta_{P2} + \vartheta_{P3} + \vartheta_{P4}).\dagger$$

If, now, we take the arbitrary line of reference parallel to a bisector of the angle formed by 12 and 34, $\vartheta_{12} + \vartheta_{34} = 0$, and the above equation becomes

$$\vartheta_t = \frac{1}{2}(\vartheta_{P1} + \vartheta_{P2} + \vartheta_{P3} + \vartheta_{P4});$$

that is, *the angle made with an* axis *of four concyclic foci of a bicircular quartic by the tangent to the quartic at any point is equal to half the sum of the angles made with the same axis by the four focal radii of the point;*‡ an *axis* of four concyclic points being understood to be any line parallel to a bisector of the angle formed

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* Proposed as a problem in the Educational Times, March 1866, and again January 1874.

† Of course, in these equations, quantities which differ by a multiple of π are considered equal.

‡ A simple algebraic proof of this theorem will be given in a subsequent article in this Journal, on bicircular quartics.

by any pair of opposite sides of the quadrangle formed by the four points; the directions of the bisectors are of course the same for every pair.

If the four real foci 1, 2, 3, 4 of a real bicircular quartic are not concyclic, they are such that ($I3$, $J4$) and ($I4$, $J3$) are foci concyclic with 1 and 2; or, in other words, two of the foci are concyclic with the antipoints of the other two. It is easily seen that the sum of the angles made with an arbitrary line by the lines joining a given point to a pair of points is not altered by the substitution of the antipoints for the original pair of points. Also, since the line joining the antipoints is perpendicular to the line joining the original pair, the "axes" of the concyclic foci are the bisectors of the angles between 12 and a perpendicular to 34. Hence if, in the case of four points such that the antipoints of one pair are concyclic with the other pair (or, what is the same thing, such that either pair are conjugate points on a diameter of a circle through the other pair; which again is equivalent to the statement that $13:23 = 14:24$) we define an axis of the four points to be a line parallel to a bisector of the angle formed by the connector of one pair and a perpendicular to the connector of the other pair, the above theorem holds for the four *real* foci, whether they are concyclic or not.

Consider, now, the function defined by the equation

$$dw = \sqrt{(w-\alpha_1)(w-\alpha_2)(w-\alpha_3)(w-\alpha_4)}\,dz.$$

When z describes a straight line, w describes a curve whose tangent makes with the path of z an angle equal to half the sum of the angles made with the axis of X by the lines drawn from w to the points α_1, α_2, α_3, α_4; or, say, the points 1, 2, 3, 4. If, then, these points are concyclic, or if they are such that $13:23 = 14:24$; and if the axes of the four points are parallel to the axes of X and Y; then, when z describes a line *parallel to either axis*, w describes one of the bicircular quartics having the four points as foci;* these quartics forming two systems orthogonal to each other. But a bicircular quartic is a closed curve not in general passing through any focus; hence w is a doubly periodic function with a real period ω and a pure imaginary period $i\omega'$. The value of ω is

$$\int ds/\sqrt{r_1 r_2 r_3 r_4}$$

*I find that Siebeck (Crelle (1860), 57, 359) showed that sn $(u+iv)$ describes a bicircular quartic when u or v is constant; and that Greenhill (Camb. Philos. Soc. Proc. (1881), 4, 77) showed the like to be true of the function inverse to $\int dw/\sqrt{(w-a_1)(w-a_2)(w-a_3)(w-a_4)}$, if the a's are concyclic. The point of view of these papers is, however, entirely different from that of the present note.

taken along a complete oval belonging to one system of the quartics; ω' is the value of the same integral taken along an oval belonging to the opposite system. And we have, incidentally, the theorem that this integral has a constant value for all ovals belonging to the same system of confocal bicircular quartics. As a limiting case, the integral of $ds/\sqrt{r_1 r_2 r_3 r_4}$, taken along the oval or the infinite branch of the circular cubic belonging to one system has the value ω; for the circular cubic belonging to the opposite system the value is ω'.

It may be noted that there is a definite distinction between the two systems of quartics; viz. if it is understood that the focal radii are drawn in a definite sense, either all *from* or all *to* the foci, then, in the quartics which correspond to the real period, the *tangent* makes with the axis of X an angle equal to half the sum of the angles made with it by these focal radii; and in the quartics which correspond to the imaginary period, the *normal* does so.

Of course, if we have

$$dw = \sqrt{(\alpha_1 - w)(\alpha_2 - w)(\alpha_3 - w)(\alpha_4 - w)}\, dz,$$

the only difference is that the two systems of quartics change places.

More generally, let

$$dw = ae^{i\lambda} \sqrt{(w - \alpha_1)(w - \alpha_2)(w - \alpha_3)(w - \alpha_4)}\, dz,$$

the points 1, 2, 3, 4 being either concyclic or such that $13:23 = 14:24$, and let the axes of the four points make the angles μ and $\mu + \frac{1}{2}\pi$ with the axis of X; then it is plain that when z moves along a line making an angle $-(\lambda + \mu)$ or $-(\lambda + \mu + \frac{1}{2}\pi)$ with the axis of X, w describes a bicircular quartic having the points 1, 2, 3, 4 as foci. Hence w has two periods, whose ratio is a pure imaginary; and the condition that one period be real and the other pure imaginary is that the axes of the four points 1, 2, 3, 4 make an angle $-\lambda$ with the axes of X and Y respectively.

The function inverse to

$$\int dw/\sqrt{a + bw + cw^2 + dw^3 + ew^4}$$

always comes under the case we have been considering *if the coefficients are real.* Namely, when the roots are real, the points 1, 2, 3, 4 are collinear and therefore concyclic; when 1 and 2 are real and 3 and 4 are conjugate imaginaries, the antipoints of 3 and 4 are collinear with 1 and 2; and when the roots are two pairs of conjugate imaginaries, the points are evidently concyclic. In all these cases, the axes of the points are evidently parallel to the axes of X and Y;

consequently the periods are one of them real and the other pure imaginary. In all these cases, too, it may be observed, the quartics have four collinear foci, the line of them being the axis of X; this has been mentioned in the first two cases, and it is true of the third case because the antipoints of the pair of points representing a pair of conjugate imaginaries lie on the axis of X.

To bring the case of

$$dw = ae^{i\lambda}\sqrt{(w-\alpha_1)(w-\alpha_2)(w-\alpha_3)}\,dz$$

under the foregoing, we must regard α_4 as having gone to infinity *along the axis of* X, so that the angle made with the axis of X by the line joining w to α_4 shall be 0, and thus not affect the construction for the tangent. In order that the four points be concyclic, 123 must be collinear; in order that the antipoints of 1 and 2 be concyclic with 3 and 4, 3 must be equidistant from 1 and 2. The paths corresponding to the periods are in this case Cartesians whose real foci are at 1, 2, 3 (the focus at infinity being ignored).

In particular,

$$dw = \sqrt{a + bw + cw^2 + dw^3}\,dz$$

falls under these heads if the coefficients are real. In this case, the three real foci are in the axis of X when the roots are real; one real focus and two imaginary foci are in the axis of X when two of the roots are imaginary; and in either event the axes of the root-points are parallel to the axes of X and Y, so that there is a real period and a pure imaginary period.

There is nothing to prevent our dropping one more factor, and considering the path corresponding to

$$dw = \sqrt{(w-\alpha_1)(w-\alpha_2)}\,dz$$

as arising from the preceding by α_3 as well as α_4 going to infinity along the axis of X. Only, in this case, one of the periods, which is $\int ds/\sqrt{r_1 r_2}$ taken along a hyperbola, becomes infinite, the denominator being an infinite of the first order.

If a function w be defined by the equation

$$z = \int^w f(w)\,dw, \tag{1}$$

$f(w)$ being *any* function with real coefficients, then the general equation of the curves described by w when z moves parallel to the axis of Y furnishes the addition-equation of the function w. For if we write

$$w = \phi(z),\ z = \phi^{-1}(w), \tag{2}$$

then, since the coefficients in $f(w)$ are real, there is evidently no loss of generality in assuming

$$\phi^{-1}(X + iY) = \text{conjugate of } \phi^{-1}(X - iY); \tag{3}$$

hence (denoting the rectangular coordinates of z by X and Y) to say that z describes a line parallel to the axis of Y is the same as to say that

$$\phi^{-1}(X + iY) + \phi^{-1}(X - iY) = \text{const.} \tag{4}$$

Therefore the equation of the curves described by w when z describes any line parallel to the axis of Y is equivalent to equation (4). Let us, then, put

$$X + iY = x, \quad X - iY = y; \tag{5}$$

equation (4) becomes

$$\phi^{-1}(x) + \phi^{-1}(y) = \text{const.} \tag{6}$$

and the equation of the w-curves becomes an equation of the form

$$F(x, y, C) = 0, \tag{7}$$

which, being equivalent to (6), is the addition-equation of the function ϕ.

Of course the connection between (6) and (7), or, say, between the differential equation $\qquad f(x)\,dx + f(y)\,dy = 0$

and the equation $\qquad F(x, y, C) = 0$

is a purely analytic one. Having once obtained the latter equation, we have no longer anything to do with the original significance of x and y, nor with the question of the reality of the coefficients in f.

What we have found, then, may be stated as follows: Let a function w be defined by the equation

$$z = \int^w f(w)\,dw,$$

the coefficients in f being literal. If, on the supposition that these coefficients are real, the general equation of the curves described by w when z moves parallel to the axis of Y, converted into an equation in x and y by the substitution

$$x = X + iY, \quad y = X - iY,$$

is

$$F(x, y, C) = 0,$$

then this equation is the solution of the equation

$$f(x)\,dx + f(y)\,dy = 0,$$

or, in other words, is the addition-equation of the function w; and the result holds, of course, whether the coefficients in f be real or imaginary.

In like manner, the equation of the curves described by w when z moves parallel to the axis of X, converted into an equation in x and y by the substitution
$$x = X + iY,\ y = X - iY,$$
is the solution of the equation
$$f(x)\,dx - f(y)\,dy = 0\,.$$
If the two sets of curves are comprised in a single equation, this is, of course, the solution of
$$f(x)\,dx \pm f(y)\,dy = 0\,.$$

For example, to find the addition-equation of the function w defined by the equation
$$z = \int^w dw / \sqrt{Aw^2 + 2Bw + C}\,.$$
We know that when z moves parallel to the axes of X and Y, the paths of w are the confocal ellipses and hyperbolas having their foci at the root-points of the quadratic under the radical sign. Expressing everything in the circular coordinates x and y $(= X + iY,\ X - iY)$ the foci are evidently given, *when* A, B, C *are real*, by the equations
$$Ax^2 + 2Bxz + Cz^2 = 0\,,\ \ Ay^2 + 2Byz + Cz^2 = 0\,,$$
the z being introduced for homogeneity. Hence the equation of the required curves is
$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0\,,$$
with such conditions among the coefficients as will make the preceding pair of equations represent the pairs of tangents from (x, z) and (y, z) respectively. These conditions are evidently
$$c_1 : g_1 : a_1 = A : -B : C\,, \quad c_1 : f_1 : b_1 = A : -B : C\,,$$
where a_1 is the minor of a in the determinant of the conic, etc. Hence we at once obtain
$$\begin{array}{ccccccccccc} a & : & b & : & c & : & f & : & g & : & h \end{array}$$
$$= AC - B^2 : AC - B^2 : C^2 - H^2 : BC - BH : BC - BH : B^2 - AH\,,$$
H being an arbitrary constant. Therefore the equation of our required system of curves, or in other words, the solution of the differential equation
$$\frac{dx}{\sqrt{Ax^2 + 2Bx + C}} \pm \frac{dy}{\sqrt{Ay^2 + 2By + C}} = 0$$
is
$$(AC - B^2)(x^2 + y^2) + 2B^2xy + 2BC(x + y) + C^2 - 2H\{Axy + B(x + y)\} - H^2 = 0\,,$$

the result holding true, of course, whether A, B, C are real or not. The result might, of course, have been otherwise obtained by writing the equation of the confocal conics in X and Y and then transforming to x and y. In particular, the integral of

$$\frac{dx}{\sqrt{1-x^2}} \pm \frac{dy}{\sqrt{1-y^2}} = 0,$$

obtained from the above by putting $A, B, C = -1, 0, 1$, is

$$x^2 + y^2 - 1 - 2Hxy + H^2 = 0,$$

whence

$$H = xy \pm \sqrt{(1-x^2)(1-y^2)},$$

which is equivalent to the formula

$$\cos(u \pm v) = \cos u \cos v \mp \sin u \sin v.$$

In the same way, to obtain the addition-equation of the function w defined by the equation

$$z = \int^w dw/\sqrt{Aw^4 + 4Bw^3 + 6Cw^2 + 4Dw + E},$$

in which the coefficients are perfectly general, we have only to find the equation, in circular coordinates, of the bicircular quartics whose foci are given by

$$Ax^4 + 4Bx^3z + 6Cx^2z^2 + 4Dxz^3 + Ez^4 = 0, \; Ay^4 + 4By^3z + 6Cy^2z^2 + 4Dyz^3 + Ez^4 = 0;$$

that is, of a system of confocal bicircular quartics, four of whose foci (not necessarily the real ones) are in a straight line. The solution of this problem (which presents no difficulty) will be contained in an article shortly to appear in this Journal, on bicircular quartics.

But it is to be observed that while the method we have been illustrating gives the *addition-equation* of the function w for the most general case, it gives the *path* of w (corresponding to a motion of z parallel to the axes) only in the special case when the coefficients are real. So far as the elliptic functions are concerned, we had found the paths of w corresponding to two periods in the more general case when the root-points of the biquadratic are concyclic or what may be called anticoncyclic; but we have not considered the path of w at all in the general case where no restriction is placed upon the position of the root-points. What we want in this case is a curve such that the tangent at any point makes with some axis an angle equal to half the sum of the angles made with the

axis by the lines joining the point to four fixed points whose position is arbitrary. I have not succeeded in geometrically determining such a curve.*

Finally, it may be worth while to make an obvious remark concerning the cases which have been considered. Since the sum of any integer multiples of two periods is a period, and conversely, the path of w will be a closed curve not only when z moves in the two mutually perpendicular directions corresponding to the periods already noticed, but also whenever z describes a straight line, the tangent of whose angle with either of these directions is the ratio of any integer multiples of ω and ω'; and in no other case. Hence we have the geometrical theorem that the oblique trajectory of a system of confocal bicircular quartics, the angle of intersection being α, is a closed curve for an infinite number of values of α; viz. whenever

$$\tan\alpha = \frac{m\int_S ds/\sqrt{r_1 r_2 r_3 r_4}}{n\int_{S'} ds/\sqrt{r_1 r_2 r_3 r_4}},$$

m and n being integers, and $\int_S$ and $\int_{S'}$ being taken along complete ovals belonging to opposite systems.

In the case of the confocal conics, the corresponding function having only one period, none of the oblique trajectories are closed curves.

* Its differential equation in circular coordinates is, however, obvious, and is immediately integrable by elliptic functions. See an article on "Some Applications of Circular Coordinates," to appear shortly in this Journal.